

# The CWKB Method of Particle Production Near Chronology Horizon

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## Abstract

In this paper we investigate the phenomenon of particle production of masses scalar field, in a model of spacetime where the chronology horizon could be formed, using the method of complex time WKB approximation (CWKB). For the purpose, we take two examples in a model of spacetime, one already discussed by Sushkov, to show that the mode of particle production near chronology horizon possesses the similar characteristic features as are found while discussing particle production in time dependent curved background. We get identical results as that obtained by Sushkov in this direction. We find, in both the examples studied, that the total number of particles remain finite at the moment of the formation of the chronology horizon.

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## 1 Introduction

The study of closed timelike curves (CTC) has gained a serious attention since its introduction by Morris, Throne, and Yurtsever [1]. Hawking [2, 3] was of the view that such closed timelike curves are not allowed in real world which we describe through the standard laws of physics. This view now runs with the name, 'Hawking chronology protection conjecture'. The principle of general relativity allows in its framework the occurrence of closed timelike curve and has been demonstrated by various authors [4, 5, 6, 7, 8, 9, 10]. The arguments behind not allowing closed timelike curves are that the renormalized energy momentum tensor diverges at the Cauchy horizon (generated by closed null geodesic) separating the regions with CTC from the region without closed causal curves. Now we have many examples where we find bounded renormalized stress-energy tensor near the chronology horizon [11, 12, 13, 14, 15]. In Gott [16] we find an elaborate discussion how CTCs play role in the early universe and could be the mother of itself i.e., the universe creates itself.

If we believe in semiclassical quantum gravity, the laws of standard physics allow the wave function of the universe to be in the description for which we have no satisfactory initial conditions. In this regime, the allowance of CTC might be a step further to understand the ‘nothing’ from where the universe emerges [16]. Now we know that the particle production is the dominant factor for the creation of the matter in the universe. So, if CTC has any role in the formation of the universe it is necessary to investigate whether the particle production near such CTC destroys it or not. Sushkov [17] tried to get an answer, considering particle production near chronology horizon and found that the total number of particles remain finite at the moment of the formation of the chronology horizon. We adopt here the similar approach but with the method of CWKB. In a general class of spacetime, finding of mode solutions and then the calculation of Bogolubov coefficients, to study particle production, is a very difficult task. The CWKB offers an way out in this direction [18, 19, 20, 21]. In this work we apply the method of CWKB to study particle production in a spacetime with a property of having CTC at distant future. For the purpose we consider a two dimensional model of spacetime akin to Sushkov [17]. The present work on the one hand substantiates the calculation of Sushkov and on the other hand allows one to use CWKB in a more general class of spacetimes with possibility of forming closed timelike curves. In this work we take two examples to elucidate our stand.

We use the units  $c = \hbar = G = 1$  throughout the paper.

## 2 Model of Spacetime

We consider a model of spacetime in which we find the chronology horizon being formed at distant future with no such behaviour at early times. We consider the metric

$$ds^2 = d\eta^2 + 2a(\eta)d\eta d\xi - [1 - a^2(\eta)]d\xi^2, \quad (1)$$

where  $a(\eta)$  is a monotonically increasing function of  $\eta$  with the following behaviour:

$$\begin{aligned} a(\eta) &\rightarrow 0 \quad \text{if } \eta \rightarrow -\infty, \\ a(\eta) &\rightarrow a_0 \quad \text{if } \eta \rightarrow +\infty, \end{aligned} \quad (2)$$

where  $a_0$  is some constant. To effectuate the occurrence of CTC we consider the strip  $\{\eta \in (-\infty, +\infty), \xi \in [0, L]\}$  on the  $\eta - \xi$  plane and assume that the points  $\xi = 0$  and  $\xi = L$  are identified i.e.,  $(\eta, 0) \equiv (\eta, L)$ . This produces a manifold with the topology of a cylinder:  $R^1 \times S^1$ . The regularity of spacetime with  $R^1 \times S^1$  topology is ensured as follows. The metric coefficients do not depend upon  $\xi$ , so the metric and its derivative takes the same values at the points  $(\eta, 0)$  and  $(\eta, L)$ . Hence the internal metrics and curvatures are identical at both lines  $\gamma^- : \xi = 0$  and  $\gamma^+ : \xi = L$ .

To study particle production it is necessary to identify the “in” and “out” regions where the particle and the antiparticle states are defined. From (1) and (2) it is evident that at  $\eta \rightarrow -\infty$ , we have exactly the Minkowski form

$$ds^2 = d\eta^2 - d\xi^2. \quad (3)$$

In the future,  $\eta \rightarrow +\infty$  we have

$$ds^2 = dt^2 - dx^2, \quad (4)$$

where the new coordinates are

$$t = \eta + a_0\xi, \quad x = \xi \quad (5)$$

Thus in out-region and in-region we can construct Minkowski-like vacuum. For any point  $(\eta, \xi)$  in  $(R^1 \times S^1)$  there are infinite number of images of points  $(\eta, \xi + L)$  in the covering space, the whole  $(\eta, \xi)$  plane. Thus we have the equivalence relation between  $R^1 \times S^1$  and the covering space as

$$(\eta, \xi + L) \equiv (\eta, \xi) \quad (6)$$

Now we obtain the equation of null curves setting  $ds^2 = 0$  in Eq.(1). We get

$$\xi + \int^\eta \frac{d\eta'}{1 + a(\eta')} = \text{const.} (\equiv C^-), \quad (7)$$

$$\xi - \int^\eta \frac{d\eta'}{1 - a(\eta')} = \text{const.} (\equiv C^+). \quad (8)$$

Here  $C^-$  describes the *left*-hand branch of the future light cone and  $C^+$  describes the *right*-hand branch. From the asymptotic properties Eq.(2), it follows from Eqs.(7) and (8) that for the null geodesics in the in-region we have

$$\eta + \xi = \text{const}, \quad \eta - \xi = \text{const}, \quad (9)$$

and in the out-region they are

$$\eta + (1 + a_0)\xi = \text{const}, \quad \eta - (1 - a_0)\xi = \text{const}. \quad (10)$$

Eqs. (10) now contain the clue for forming the chronology horizon. As  $\eta$  gets larger the *right*-hand branch of the future light cone rotates rightward and ultimately if  $a_0 \rightarrow 0$ , it coincides with the  $\xi$ -axis so that we get  $\eta = \text{constant}$ . Now from Eq. (6) we find that the curve in the  $\xi$  direction is closed. If this occurs at the moment  $\eta = \eta^*$  such that  $C^+$  branch becomes horizontal, then the closed null curves appear in our model. We now say that a time machine is being formed at this moment of time. If  $\eta^* = \infty$  ( $a_0 = 1$ ), the time machine is formed in the infinitely far future. Otherwise, if we have  $\eta > \eta^*$ , the closed line  $\eta = \text{constant}$  lies inside of the light cone. For more details, the reader is referred to ref.[17]. The discussion exemplifies the occurrence of CTC in the spacetime defined by the metric (1).

We now investigate the particle creation in such a spacetime.

### 3 Particle Creation

We consider the massless scalar field equation

$$\square \phi = 0 \quad (11)$$

in the metric given by Eq.(1). In this metric the wave equation reads

$$\left[ (1 - a^2) \partial_\eta^2 + 2a \partial_\eta \partial_\xi - \partial_\xi^2 - 2a' \partial_\eta + a' \partial_\xi \right] \phi(\eta, \xi) = 0, \quad (12)$$

where  $\partial_\eta = \partial/\partial\eta$ ,  $\partial_\xi = \partial/\partial\xi$ , and a prime denotes the derivative on  $\eta$ ,  $a' = da/d\eta$ . In the in-region  $a(\eta) \rightarrow 0$  and  $a'(\eta) \rightarrow 0$  Eq. (12) reduces to the form

$$\left[ \partial_\eta^2 - \partial_\xi^2 \right] \phi(\eta, \xi) = 0. \quad (13)$$

The complete set of solutions of this equation is

$$\phi_n^{\pm, in} = D_n^{(in)} e^{ik_n \xi} e^{\mp i \omega \eta}, \quad (14)$$

where  $\omega = |k_n|$  with  $k_n = \frac{2\pi n}{L}$ ,  $n = \pm 1, \pm 2, \dots$  being determined from the boundary condition

$$\phi(\eta, \xi + L) = \phi(\eta, \xi). \quad (15)$$

From the normalization condition

$$(\phi_n, \phi_{n'}) = -i \int_0^L d\xi \left( \phi_n \frac{\partial \phi_{n'}^*}{\partial \eta} - \frac{\partial \phi_n}{\partial \eta} \phi_{n'}^* \right)_{\eta=const.} = \delta_{nn'} \quad (16)$$

we get

$$D_n^{(in)} = \frac{1}{\sqrt{4\pi|n|}}. \quad (17)$$

To obtain the mode solution in the out region, we have  $a(\eta) \rightarrow a_0$  and  $a'(\eta) \rightarrow 0$  so that Eq.(12) reduces to

$$\left[ (1 - a_0^2) \partial_\eta^2 + 2a_0 \partial_\eta \partial_\xi - \partial_\xi^2 \right] \phi(\eta, \xi) = 0. \quad (18)$$

We find the solution as before as

$$\phi_n^{(\pm, out)} = D_n^{(out)} e^{ik_n \xi} e^{ik_n a_0 \beta^{-1} \eta} e^{\mp i \omega \beta^{-1} \eta}, \quad (19)$$

with  $\beta = 1 - a_0^2$  and

$$D_n^{(out)} = \sqrt{\frac{\beta}{4\pi|n|}}. \quad (20)$$

It is evident from Eqs.(14) and (19) that  $\omega_{in} = \omega \neq \omega_{out} = (\omega \pm k_n a_0) \beta^{-1}$  so that there is possibility of particle production as the particles evolve from the in-vacuum to out-vacuum. Let us introduce

$$\phi_n(\eta, \xi) = v(\eta) \exp \left( - \int^\eta \frac{a(ik_n - a')}{1 - a^2} d\eta \right) \exp(ik_n \xi) \quad (21)$$

in Eq.(12) to obtain

$$v'' + \Omega^2(\eta)v = 0, \quad (22)$$

where

$$\Omega^2(\eta) = \frac{k_n^2 + a'^2 + a''a(1 - a^2)}{(1 - a^2)^2} \quad (23)$$

We solve Eq.(22) only for modes for which

$$k_n^2 \gg a'^2, \quad k_n^2 \gg a''a(1 - a^2) \quad (24)$$

i.e., we consider only those modes whose wavelength is much less than a typical scale of variation of the function  $a(\eta)$ . We now consider two cases:

$$(I) \quad a(\eta) = \frac{a_0^2}{1 + \exp(-2\gamma\eta)}, \quad (25)$$

$$(II) \quad a(\eta) = \frac{1}{2}a_0^2(1 + \tanh\gamma\eta) \quad (26)$$

The case (II) has already been discussed by Sushkov [17]. We will now use the method of CWKB to obtain the number of created particles near the chronology horizon where  $a_0 \rightarrow 1$ . According to CWKB, the boundary conditions for particle production is taken as

$$v(\eta \rightarrow -\infty) \sim e^{iS(\eta, \eta_0)}, \quad (27)$$

$$v(\eta \rightarrow +\infty) \sim e^{iS(\eta, \eta_0)} + R e^{-iS(\eta, \eta_0)}. \quad (28)$$

Here we have neglected the WKB pre-exponential factor for convenience and

$$S(\eta, \eta_0) = \int_{\eta_0}^{\eta} \Omega(\eta) d\eta \quad (29)$$

and the reflection amplitude  $R$  that accounts particle production is given by

$$R = \frac{-ie^{2iS(\eta_1, \eta_0)}}{\sqrt{1 + e^{2iS(\eta_1, \eta_2)}}} \quad (30)$$

where the turning points are given by the condition  $\Omega(\eta_{1,2}) = 0$ . In (30) the denominator takes into account the repeated reflections between the turning points  $\eta_1$  and  $\eta_2$ . In case of single turning point, we have

$$R = -ie^{2iS(\eta_1, \eta_0)} \quad (31)$$

In above equations  $\eta_0$  is an arbitrary real point and does not affect the magnitude of  $R$ . The derivation of Eq.(30) can be found in our earlier works [18,19,20]. For case (I) we are to evaluate the integral

$$I = S(\eta, \eta_0) = k_n \int_{\eta_0}^{\eta} \frac{d\eta}{1 - \frac{a_0^2}{1 + \exp(-2\gamma\eta)}}. \quad (32)$$

We avoid the  $\eta_0$  limit as it would give a real contribution to  $I$  and hence does not contribute to  $|R|$  in Eq.(30). We now substitute  $\tau = \exp(-2\gamma\eta)$  and take  $1 - a_0^2 = \beta$  in the above integral and find

$$I = -\frac{k_n}{2\gamma} \int^\eta \frac{(\tau + 1)d\tau}{(\beta\tau + 1)\tau}. \quad (33)$$

We now have the turning point at  $\tau = -1$  i.e., at complex  $\eta = \pm i\pi/2\gamma$ . After evaluation we get

$$I = -\frac{k_n}{2\gamma} \left[ \ln \tau + \frac{1 - \beta}{\beta} (\ln \tau - \ln(\beta + \tau)) \right] \quad (34)$$

To evaluate  $I$  we should be careful. For  $\beta \neq 0$ , only the first and second term contributes but for  $\beta \rightarrow 0$ , all the three terms contribute and we get

$$I = -\frac{k_n}{2\gamma} i\pi. \quad (35)$$

Using the expression of  $R$ , we now get

$$|R|^2 = e^{-\frac{\pi k_n}{\gamma}} \quad (36)$$

The Bogolubov coefficient  $\beta_n$  is now evaluated as [18]

$$\begin{aligned} |\beta_n|^2 &= \frac{|R|^2}{|T|^2} \\ &= \frac{|R|^2}{(1 - |R|^2)}, \end{aligned} \quad (37)$$

so that

$$|\beta_n|^2 = \frac{e^{-\pi k_n/\gamma}}{1 - e^{-\pi k_n/\gamma}} \quad (38)$$

Let us now consider the case II. Here we are to evaluate the integral

$$I = k_n \int \frac{d\eta}{1 - \frac{1}{2}a_0^2(1 + \tanh\gamma\eta)} \quad (39)$$

As before we put  $t = e^{2\gamma\eta}$  and get

$$I = \frac{k_n}{2\gamma} \int \frac{dt(t+1)}{(\beta t + 1)t}, \quad (40)$$

where  $\beta = 1 - a_0^2$ . After evaluating the integral we find

$$I = \frac{k_n}{2\gamma} \ln t - \frac{k_n(\beta - 1)}{2\gamma\beta} \ln(\beta t + 1). \quad (41)$$

Here the turning point is again at  $t = -1$  so that

$$I(t = -1) = \frac{i\pi k_n}{2\gamma} + \delta \quad (42)$$

Here  $\delta$  term comes from the real point  $\eta_0$  as well as from the real part contribution of the above integral and it does not contribute in the expression of  $|R|$ . Hence in both the cases we find the number of particles produced in the mode  $k_n$  is

$$N_n = |\beta_n|^2 = \frac{e^{-\pi k_n/\gamma}}{1 - e^{-\pi k_n/\gamma}}. \quad (43)$$

We may now conclude that the total number of particles  $N = \sum_n N_n$  will be finite because the spectrum (43) is exponentially decreasing. In obtaining (43) we have used the condition (24) which when evaluated reduces to

$$k_n^2 \gg n \frac{a_0^2}{T^2}, \quad (44)$$

where  $T = (2\gamma)^{-1}$  gives the typical time variation of the function  $a(\eta)$  from one asymptotical value to another one and  $n$  is a factor of  $O(1)$ .

## 4 conclusion

The chronology horizon is formed when  $\beta = 1 - a_0^2 \rightarrow 0$  i.e., at the moment of time  $\eta^* < \infty$  if  $a_0 > 1$ . In the case  $a_0 = 1$ , closed lightlike curves are formed in the infinitely far future. In our work we obtain almost identical results as that obtained by Sushkov and substantiate the conclusion that the phenomenon of particle creation could not prevent the formation of a time machine. For  $a_0 < 1$  we have no causal pathologies.

While discussing the particle production in curved spacetime we noticed that in CWKB we do not require to know the exact solutions of the problem in question. The vacua are the WKB vacua as defined by Parker. As the particles evolve from the in-vacuum to the out-vacuum they find  $|0\rangle_{in} \neq |0\rangle_{out}$ . The reasons for such a change is that the particle moves into the Euclidean vacuum; in other words, it encounters complex  $\eta$  or  $t$  plane and causes the instability of vacuum causing particle production. From the discussion it is clear that since  $\Omega \neq 0$  in the real time plane we are basically studying the over the barrier reflection. As  $(\frac{da}{d\eta})_0 \sim \gamma$  is in the expression in the denominator of  $Im I$ , a sharp rise is thus needed so that  $exp(-\pi k_n/\gamma)$  remains small. What we observe that the particles so produced still survive when the chronology horizon is formed in distant future. It is therefore essential to investigate how would this energy density of produced particles affect the spacetime metric and the formation of chronology horizon.

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